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Strichartz estimates for the magnetic Schrödinger equation with potentials V of critical decay

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ABSTRACT

We study the Strichartz estimates for the magnetic Schrödinger equation in dimension $n \geq 3$. More specifically, for all Schrödinger admissible pairs (r, q), we establish the estimate

$$\|e^{itH}f\|_{L^q_t(\mathbb{R};L^r_x(\mathbb{R}^n))} \leq C_{n,r,q,H}\|f\|_{L^2(\mathbb{R}^n)}$$

when the operator $H = -\Delta_A + V$ satisfies suitable conditions. In the purely electric case $A \equiv 0$, we extend the class of potentials V to the Fefferman–Phong class. In doing so, we apply a weighted estimate for the Schrödinger equation developed by Ruiz and Vega. Moreover, for the endpoint estimate of the magnetic case in $\mathbb{R}^3$, we investigate an equivalence

$$\|H^{\frac{1}{4}}f\|_{L^r(\mathbb{R}^3)} \approx C_{H,r}\left\|(-\Delta)^{\frac{1}{4}}f\right\|_{L^r(\mathbb{R}^3)}$$

and find sufficient conditions on H and r for which the equivalence holds.

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1. Introduction

Consider the Cauchy problem of the magnetic Schrödinger equation in $\mathbb{R}^{n+1}$ $(n \geq 3)$:

$$\begin{cases} i\partial_t u - Hu = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}. \end{cases} \tag{1.1}$$

Here, $\mathcal{S}$ is the Schwartz class, and H is the electromagnetic Schrödinger operator

$$H = -\nabla_A^2 + V(x), \quad \nabla_A = \nabla - iA(x),$$

where $A = (A^1, A^2, \cdots, A^n) : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$. The *magnetic field* B is defined by

$$B = DA - (DA)^T \in \mathcal{M}_{n\times n},$$

where $(DA)_{ij} = \partial_{x_i}A^j$, $(DA)^T$ denotes the transpose of DA, and $\mathcal{M}_{n\times n}$ is the space of $n \times n$ real matrices. In dimension $n = 3$, B is determined by the cross product with the vector field curl$A$:

$$Bv = \text{curl}A \times v \quad (v \in \mathbb{R}^3).$$

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In this paper, we consider the Strichartz type estimate

$$\|u\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))} \le C_{n,r,q,H}\|f\|_{L^2(\mathbb{R}^n)}, \tag{1.2}$$

where $u = e^{itH}f$ is the solution to problem (1.1) with solution operator e^{itH}, and study some conditions on A, V and pairs (r, q) for which the estimate holds.

For the unperturbed case of (1.1) that $A \equiv 0$ and $V \equiv 0$, Strichartz [33] proved the inequality

$$\|e^{it\Delta}f\|_{L^{\frac{2(n+2)}{n}}(\mathbb{R}^{n+1})} \le C_n\|f\|_{L^2(\mathbb{R}^n)},$$

where $e^{it\Delta}$ is the solution operator given by

$$e^{it\Delta}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi+it|\xi|^2}\widehat{f}(\xi)d\xi.$$

Later, Keel and Tao [23] generalized this inequality to the following:

$$\|e^{it\Delta}f\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))} \le C_{n,r,q}\|f\|_{L^2(\mathbb{R}^n)} \tag{1.3}$$

holds if and only if (r, q) is a Schrödinger admissible pair; that is, $r, q \ge 2$, $(r, q) \ne (\infty, 2)$, and $\frac{n}{r} + \frac{2}{q} = \frac{n}{2}$.

In the purely electric case of (1.1) that $A \equiv 0$, the decay $|V(x)| \sim 1/|x|^2$ has been known to be critical for the validity of the Strichartz estimate. It was shown by Goldberg et al. [21] that for each $\epsilon > 0$, there is a counterexample of $V = V_\epsilon$ with $|V(x)| \sim |x|^{-2+\epsilon}$ for $|x| \gg 1$ such that the estimate fails to hold. In a positive direction, Rodnianski and Schlag [26] proved

$$\|u\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))} \le C_{n,r,q,V}\|f\|_{L^2(\mathbb{R}^n)} \tag{1.4}$$

for non-endpoint admissible pairs (r, q) (i.e., $q > 2$) with almost critical decay $|V(x)| \lesssim 1/(1+|x|)^{2+\epsilon}$. However, Burq et al. [3] established (1.4) for critical decay $|V(x)| \lesssim 1/|x|^2$ with some technical conditions on V but the endpoint case included. Other than these, there have been many related positive results; see, e.g., [20], [18], [1], [15] and [2].

In regard to the purely electric case, the following is the first main result of this paper whose proof is given in Section 2.

Theorem 1.1. *Let $n \ge 3$ and $A \equiv 0$. Then there exists a constant $c_n > 0$, depending only on n, such that for any $V \in \mathcal{F}^p$ ($\frac{n-1}{2} < p < \frac{n}{2}$) satisfying*

$$\|V\|_{\mathcal{F}^p} \le c_n, \tag{1.5}$$

estimate (1.4) holds for all $\frac{n}{r} + \frac{2}{q} = \frac{n}{2}$ and $q > 2$. Moreover, if $V \in L^{\frac{n}{2}}$ in addition, then (1.4) holds for the endpoint case $(r, q) = (\frac{2n}{n-2}, 2)$.

Here, $\mathcal{F}^p$ is the Fefferman–Phong class with norm

$$\|V\|_{\mathcal{F}^p} = \sup_{r>0,\, x_0\in\mathbb{R}^n} r^2\left(\frac{1}{r^n}\int_{B_r(x_0)}|V(x)|^p dx\right)^{\frac{1}{p}} < \infty,$$

which is closed under translation. From the definition of $\mathcal{F}^p$, we directly get $L^{\frac{n}{2},\infty} \subset \mathcal{F}^p$ for all $p < \frac{n}{2}$. Thus the class $\mathcal{F}^p$ ($p < \frac{n}{2}$) clearly contains the potentials of critical decay $|V(x)| \lesssim 1/|x|^2$. Moreover, $\mathcal{F}^p$ ($p < \frac{n}{2}$) is strictly larger than $L^{\frac{n}{2},\infty}$. For instance, if the potential function

$$V(x) = \phi\left(\frac{x}{|x|}\right)|x|^{-2}, \quad \phi \in L^p(S^{n-1}), \quad \frac{n-1}{2} < p < \frac{n}{2}, \tag{1.6}$$

then V need not belong to $L^{\frac{n}{2},\infty}$, but $V \in \mathcal{F}^p$.

According to Theorem 1.1, for the non-endpoint case, we do not need any other conditions on V but its quantitative bound (1.5), so that we can extend and much simplify the known results for potentials $|V(x)| \sim 1/|x|^2$ (e.g., $\phi \in L^\infty(S^{n-1})$ in (1.6)), mentioned above. To prove this, we use a weighted estimate developed by Ruiz and Vega [28]. We remark that our proof follows an approach different from those used in the previous works.

Unfortunately, for the endpoint case, we need an additional condition that $V \in L^{\frac{n}{2}}$. Although $L^{\frac{n}{2}}$ dose not contain the potentials of critical decay, it still includes those of *almost* critical decay $|V(x)| \lesssim \phi(\frac{x}{|x|})\min(|x|^{-(2+\epsilon)}, |x|^{-(2-\epsilon)})$, $\phi \in L^{\frac{n}{2}}(S^{n-1})$.

In case of dimension $n = 3$, we can find a specific bound for V, which plays the role of c_n in Theorem 1.1. We state this as the second result of the paper.

Theorem 1.2. *If $n = 3$, $A \equiv 0$ and $\|V\|_{L^{3/2}} < 2\pi^{1/3}$, then estimate (1.4) holds for all $\frac{3}{r}+\frac{2}{q} = \frac{3}{2}$ and $q \geq 2$.*

To prove this, we use the best constant of the Stein–Tomas restriction theorem in $\mathbb{R}^3$, obtained by Foschi [16], and apply it to an argument of Ruiz and Vega [28].

Next, we consider the general (magnetic) case that A or V can be different from zero. In this case, the Coulomb decay $|A(x)| \sim 1/|x|$ seems critical. (In [14], there is a counterexample for $n \geq 3$. The case $n = 2$ is still open.) In an early work of Stefanov [32], estimate (1.2) for $n \geq 3$ was proved, that is,

$$\|e^{itH}f\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))} \leq C_{n,r,q,H}\|f\|_{L^2(\mathbb{R}^n)} \tag{1.7}$$

for all Schrödinger admissible pairs (r, q) under some smallness assumptions on the potentials A and V. Later, for potentials of almost critical decay $|A(x)| \lesssim 1/|x|^{1+\epsilon}$ and $|V(x)| \lesssim 1/|x|^{2+\epsilon}$ ($|x| \gg 1$), D'Ancona et al. [7] established (1.7) for all Schrödinger admissible pairs (r, q) in $n \geq 3$, except the endpoint case $(n, r, q) = (3, 6, 2)$, under some technical conditions on A and V. Also, there have been many related positive results; see, e.g., [17], [11], [6], [24], [12], [19] and [13]. Despite all these results, there has been no known positive result on the estimate in case of potentials A of critical decay even in the case $V \equiv 0$.

Regarding the general case, we state the last result of the paper whose proof is provided in Section 4.

Theorem 1.3. *Let $n \geq 3$, $A, V \in C^1_{loc}(\mathbb{R}^n\backslash\{0\})$ and $\epsilon > 0$. Assume that the operator $\Delta_A = -(\nabla - iA)^2$ and $H = \Delta_A + V$ are self-adjoint and positive on L^2 and that*

$$\|V_-\|_K = \sup_{x\in\mathbb{R}^n}\int \frac{|V_-(y)|}{|x-y|^{n-2}}dy < \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}-1)}. \tag{1.8}$$

Assume also that there is a constant $C_\epsilon > 0$ such that A and V satisfy the almost critical decay condition

$$|A(x)|^2 + |V(x)| \leq C_\epsilon \min\left(\frac{1}{|x|^{2-\epsilon}}, \frac{1}{|x|^{2+\epsilon}}\right) \tag{1.9}$$

and the Coulomb gauge condition

$$\nabla \cdot A = 0. \tag{1.10}$$

Lastly, for the trapping component of B *as* $B_\tau(x) = (x/|x|) \cdot B(x)$, *assume that*

$$\int_0^\infty \sup_{|x|=r} |x|^3 \left|B_\tau(x)\right|^2 dr + \int_0^\infty \sup_{|x|=r} |x|^2 \left|\left(\partial_r V(x)\right)_+\right| dr < \frac{1}{M} \quad \text{if } n = 3, \tag{1.11}$$

for some $M > 0$, *and that*

$$\left\| |x|^2 B_\tau(x) \right\|_{L^\infty}^2 + 2 \left\| |x|^3 \left(\partial_r V(x)\right)_+ \right\|_{L^\infty} < \frac{2(n-1)(n-3)}{3} \quad \text{if } n \geq 4. \tag{1.12}$$

Only for $n = 3$, *we also assume the boundness of the imaginary power of* H:

$$\|H^{iy}\|_{BMO \to BMO_H} \leq C(1+|y|)^{3/2}. \tag{1.13}$$

Then we have

$$\|e^{itH} f\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^n))} \leq C_{n,r,q,H,\epsilon} \|f\|_{L^2(\mathbb{R}^n)}, \quad \frac{n}{r} + \frac{2}{q} = \frac{n}{2} \quad \text{and} \quad q \geq 2. \tag{1.14}$$

Note that this result covers the endpoint case $(n, r, q) = (3, 6, 2)$; but the conclusions for the other cases are the same as in [7]. Here, $V_\pm$ denote the positive and negative parts of V, respectively; that is, $V_+ = \max\{V, 0\}$ and $V_- = \max\{-V, 0\}$. Also, we say that a function V is of Kato class if

$$\|V\|_K := \sup_{x \in \mathbb{R}^n} \int \frac{|V(y)|}{|x-y|^{n-2}} dy < \infty,$$

and Γ in (1.8) is the gamma function, defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$. The last condition (1.13) for $n = 3$ may seem a bit technical but not be artificial. For instance, by Lemma 6.1 in [8], we know that $\|H^{iy}\|_{L^\infty \to BMO_H} \leq C(1+|y|)^{3/2}$ using only (1.8). Also, there are many known sufficient conditions to extend such an estimate to $BMO \to BMO$, like the translation invariant operator [25]. For the definition and some basic properties of BMO_H space, see Section 3.

The rest of the paper is organized as follows. In Section 2, we prove Theorems 1.1 and 1.2. An equivalence of norms regarding H and $-\Delta$ in $\mathbb{R}^3$ is investigated in Section 3. Finally, in Section 4, Theorem 1.3 is proved.

2. The case $A \equiv 0$: proof of Theorems 1.1 and 1.2

In this section, the proof of Theorems 1.1 and 1.2 is provided. Let $n \geq 3$, and consider the purely electric Schrödinger equation in $\mathbb{R}^{n+1}$:

$$\begin{cases} i\partial_t u + \Delta u = V(x)u, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}. \end{cases} \tag{2.1}$$

By Duhamel's principle, we have a formal solution to problem (2.1) given by

$$u(x,t) = e^{itH} f(x) = e^{it\Delta} f(x) - i \int_0^t e^{i(t-s)\Delta} V(x) e^{isH} f ds.$$

From the standard Strichartz estimate (1.3), there holds

$$\|e^{itH}f\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))} \leq C_{n,r,q}\|f\|_{L^2(\mathbb{R}^n)} + \Big\| \int_0^t e^{i(t-s)\Delta} V(x)e^{isH} f ds\Big\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))}$$

for all Schrödinger admissible pairs (r, q). Thus it is enough to show that

$$\Big\| \int_0^t e^{i(t-s)\Delta} V(x)e^{isH} f ds\Big\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))} \leq C_{n,r,q,V}\|f\|_{L^2(\mathbb{R}^n)} \tag{2.2}$$

for all Schrödinger admissible pairs (r, q).

By the duality argument, estimate (2.2) is equivalent to

$$\int_{\mathbb{R}}\int_0^t \Big\langle e^{i(t-s)\Delta}\big(V(x)e^{isH}f\big), G(\cdot,t)\Big\rangle_{L_x^2} dsdt \leq C\|f\|_{L^2(\mathbb{R}^n)}\|G\|_{L_t^{q'}(\mathbb{R};L_x^{r'}(\mathbb{R}^n))}.$$

Now, we consider the left-hand side of this inequality. Commuting the operator and integration, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^t \Big\langle e^{i(t-s)\Delta}\big(V(x)e^{isH}f\big), G(\cdot,t)\Big\rangle_{L_x^2} dsdt \\
&= \int_{\mathbb{R}}\int_0^t \Big\langle V(x)e^{isH}f, e^{-i(t-s)\Delta}G(\cdot,t)\Big\rangle_{L_x^2} dsdt \\
&= \int_{\mathbb{R}} \Big\langle V(x)e^{isH}f, \int_s^\infty e^{-i(t-s)\Delta}G(\cdot,t)dt\Big\rangle_{L_x^2} ds.
\end{aligned}$$

By Hölder's inequality, we have

$$\int_{\mathbb{R}} \Big\langle V(x)e^{isH}f, \int_s^\infty e^{-i(t-s)\Delta}G(\cdot,t)dt\Big\rangle_{L_x^2} ds \leq \big\|e^{isH}f\big\|_{L_{x,s}^2(|V|)}\Big\|\int_s^\infty e^{-i(t-s)\Delta}G(\cdot,t)dt\Big\|_{L_{x,s}^2(|V|)}.$$

Thanks to [28, Theorem 3], for any $\frac{n-1}{2} < p < \frac{n}{2}$, we have

$$\big\|e^{itH}f\big\|_{L_{x,t}^2(|V|)} \leq C_n\|V\|_{\mathcal{F}^p}^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^n)} \tag{2.3}$$

if condition (1.5) holds for some suitable constant c_n. More specifically, by Propositions 2.3 and 4.2 in [28], we have

$$\begin{aligned}
\|e^{itH}f\|_{L_{x,t}^2(|V|)} &\leq \big\|e^{it\Delta}f\big\|_{L_{x,t}^2(|V|)} + \Big\|\int_0^t e^{i(t-s)\Delta}V(x)e^{isH}f ds\Big\|_{L_{x,t}^2(|V|)} \\
&\leq C_1\|V\|_{\mathcal{F}^p}^{\frac{1}{2}}\|f\|_{L^2} + C_2\|V\|_{\mathcal{F}^p}\big\|V(x)e^{itH}f\big\|_{L_{x,t}^2(|V|^{-1})} \\
&= C_1\|V\|_{\mathcal{F}^p}^{\frac{1}{2}}\|f\|_{L^2} + C_2\|V\|_{\mathcal{F}^p}\|e^{itH}f\|_{L_{x,t}^2(|V|)}.
\end{aligned}$$

Thus, if $\|V\|_{\mathcal{F}^p} \leq 1/(2C_2) =: c_n$, we get

$$\big\|e^{itH}f\big\|_{L_{x,t}^2(|V|)} \leq C_1\|V\|_{\mathcal{F}^p}^{\frac{1}{2}}\|f\|_{L^2} + \frac{1}{2}\|e^{itH}f\|_{L_{x,t}^2(|V|)},$$

and this implies (2.3) by setting $C_n = 2C_1$. As a result, we can reduce (2.2) to

$$\Big\|\int_t^\infty e^{i(t-s)\Delta}G(\cdot,s)ds\Big\|_{L_{x,t}^2(|V|)} \leq C_{n,r,q,V}\|G\|_{L_t^{q'}(\mathbb{R};L_x^{r'}(\mathbb{R}^n))}. \tag{2.4}$$

It now remains to establish (2.4). First, from [28, Proposition 2.3] and the duality of Keel–Tao's result (1.3), we know that

$$\begin{aligned}\Big\| \int_{\mathbb{R}} e^{i(t-s)\Delta} G(\cdot,s)ds \Big\|_{L^2_{x,t}(|V|)} &= \Big\| e^{it\Delta} \int_{\mathbb{R}} e^{-is\Delta} G(\cdot,s)ds \Big\|_{L^2_{x,t}(|V|)} \\ &\le C_n \|V\|^{\frac{1}{2}}_{\mathcal{F}^p} \Big\| \int_{\mathbb{R}} e^{-is\Delta} G(\cdot,s)ds \Big\|_{L^2_x} \\ &\le C_{n,r,q} \|V\|^{\frac{1}{2}}_{\mathcal{F}^p} \|G\|_{L_t^{q'} L_x^{r'}} \end{aligned} \tag{2.5}$$

for all Schrödinger admissible pairs (r,q). In turn, (2.5) implies

$$\Big\| \int_{-\infty}^{t} e^{i(t-s)\Delta} G(\cdot,s)ds \Big\|_{L^2_{x,t}(|V|)} \le C_{n,r,q} \|V\|^{\frac{1}{2}}_{\mathcal{F}^p} \|G\|_{L_t^{q'}(\mathbb{R};L_x^{r'}(\mathbb{R}^n))} \tag{2.6}$$

by the Christ–Kiselev lemma [5] for $q > 2$. Combining (2.5) with (2.6), we directly get (2.4) for $q > 2$. Next, for the endpoint case $(r,q) = (\frac{2n}{n-2}, 2)$, we have

$$\begin{aligned}\Big\| \int_{-\infty}^{t} e^{i(t-s)\Delta} G(\cdot,s)ds \Big\|_{L^2_{x,t}(|V|)} &\le \|V\|^{\frac{1}{2}}_{L_x^{\frac{n}{2}}} \Big\| \int_{-\infty}^{t} e^{i(t-s)\Delta} G(\cdot,s)ds \Big\|_{L_t^2(\mathbb{R};L_x^{\frac{2n}{n-2}}(\mathbb{R}^n))} \\ &\le C_n \|V\|^{\frac{1}{2}}_{L_x^{\frac{n}{2}}} \|G\|_{L_t^2(\mathbb{R};L_x^{\frac{2n}{n+2}}(\mathbb{R}^n))} \end{aligned} \tag{2.7}$$

from Hölder's inequality in x with the inhomogeneous Strichartz estimates by Keel-Tao. Observe now that (2.7) implies (2.4) when $q = 2$ under the assumption $V \in L_x^{\frac{n}{2}}$.

The proof of Theorem 1.1 is now complete.

Now, we will find a suitable constant in Theorem 1.2. For this, we refine estimate (2.3) based on an argument in [28]. We recall the Fourier transform in $\mathbb{R}^n$, defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x)dx,$$

and its basic properties

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)d\xi \quad \text{and} \quad \|f\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{\frac{n}{2}}} \|\widehat{f}\|_{L^2(\mathbb{R}^n)}.$$

Thus, we can express $e^{it\Delta}f$ using the polar coordinates with $r^2 = \lambda$ as follows:

$$\begin{aligned} e^{it\Delta}f &= \frac{1}{(2\pi)^n} \int_0^\infty e^{itr^2} \int_{S_r^{n-1}} e^{ix\cdot\xi} \widehat{f}(\xi) d\sigma_r(\xi) dr \\ &= \frac{1}{2(2\pi)^n} \int_0^\infty e^{it\lambda} \int_{S^{n-1}_{\sqrt{\lambda}}} e^{ix\cdot\xi} \widehat{f}(\xi) d\sigma_{\sqrt{\lambda}}(\xi) \lambda^{-\frac{1}{2}} d\lambda. \end{aligned}$$

Take F as

$$F(\lambda) = \int_{S^{n-1}_{\sqrt{\lambda}}} e^{ix\cdot\xi} \widehat{f}(\xi) d\sigma_{\sqrt{\lambda}}(\xi) \lambda^{-\frac{1}{2}}$$

if $\lambda \geq 0$ and $F(\lambda) = 0$ if $\lambda < 0$. Then, by Plancherel's theorem in t, we get

$$\begin{aligned}\|e^{it\Delta}f\|^2_{L^2_{x,t}(|V|)} &= \frac{2\pi}{4(2\pi)^{2n}}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}}|F(\lambda)|^2 d\lambda\right)|V(x)|dx \\ &= \frac{\pi}{2(2\pi)^{2n}}\int_{\mathbb{R}^n}\left(\int_0^\infty\left|\int_{S^{n-1}_{\sqrt{\lambda}}} e^{ix\cdot\xi}\widehat{f}(\xi)d\sigma_{\sqrt{\lambda}}(\xi)\right|^2\lambda^{-1}d\lambda\right)|V(x)|dx \\ &= \frac{\pi}{(2\pi)^{2n}}\int_0^\infty\left(\int_{\mathbb{R}^n}\left|\int_{S^{n-1}_r} e^{ix\cdot\xi}\widehat{f}(\xi)d\sigma_r(\xi)\right|^2|V(x)|dx\right)r^{-1}dr.\end{aligned}$$

Now, we consider the $n = 3$ case and apply the result on the best constant of the Stein–Tomas restriction theorem in $\mathbb{R}^3$ obtained by Foschi [16]. That is,

$$\|\widehat{fd\sigma}\|_{L^4(\mathbb{R}^3)} \leq 2\pi\|f\|_{L^2(S^2)}$$

where

$$\widehat{fd\sigma}(x) = \int_{S^{n-1}} e^{-ix\cdot\xi}f(\xi)d\sigma(\xi).$$

Interpolating this with a trivial estimate

$$\|\widehat{fd\sigma}\|_{L^\infty(\mathbb{R}^3)} \leq \|f\|_{L^1(S^2)} \leq \sqrt{4\pi}\|f\|_{L^2(S^2)},$$

we get

$$\|\widehat{fd\sigma}\|_{L^6(\mathbb{R}^3)} \leq 2^{1/6}(2\pi)^{5/6}\|f\|_{L^2(S^2)}.$$

By Hölder's inequality, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^3}\left|\int_{S^2} e^{ix\cdot\xi}\widehat{f}(\xi)d\sigma(\xi)\right|^2|V(x)|dx\right) &\leq \left\|\int_{S^2} e^{ix\cdot\xi}\widehat{f}(\xi)d\sigma(\xi)\right\|^2_{L^6}\|V\|_{L^{3/2}} \\ &\leq 2^{1/3}(2\pi)^{5/3}\|V\|_{L^{3/2}}\|\widehat{f}\|^2_{L^2(S^2)}.\end{aligned}$$

So we get

$$\begin{aligned}\|e^{it\Delta}f\|^2_{L^2_{x,t}(|V|)} &\leq \frac{\pi}{(2\pi)^6}2^{1/3}(2\pi)^{5/3}\|V\|_{L^{3/2}}\|\widehat{f}\|^2_{L^2} \\ &= \frac{1}{2\pi^{1/3}}\|V\|_{L^{3/2}}\|f\|^2_{L^2}.\end{aligned}$$

By the argument as in the proof of Theorem 1.1, we have

$$\begin{aligned}\|e^{itH}f\|_{L^2_{x,t}(|V|)} &\leq \|e^{it\Delta}f\|_{L^2_{x,t}(|V|)} + \left\|\int_0^t e^{i(t-s)\Delta}V(x)e^{isH}fds\right\|_{L^2_{x,t}(|V|)} \\ &\leq \frac{1}{\sqrt{2}\pi^{1/6}}\|V\|^{\frac{1}{2}}_{L^{3/2}}\|f\|_{L^2} + \frac{1}{2\pi^{1/3}}\|V\|_{L^{3/2}}\left\|V(x)e^{itH}f\right\|_{L^2_{x,t}(|V|^{-1})} \\ &= \frac{1}{\sqrt{2}\pi^{1/6}}\|V\|^{\frac{1}{2}}_{L^{3/2}}\|f\|_{L^2} + \frac{1}{2\pi^{1/3}}\|V\|_{L^{3/2}}\|e^{itH}f\|_{L^2_{x,t}(|V|)}.\end{aligned}$$

Thus, if $\|V\|_{L^{3/2}} < 2\pi^{1/3}$, then

$$\|e^{itH}f\|_{L^2_{x,t}(|V|)} \leq C_V\|f\|_{L^2}. \tag{2.8}$$

Using (2.8) instead of (2.3) in that argument, the proof of Theorem 1.2 is complete.

3. The equivalence of two norms involving H and $-\Delta$ in $\mathbb{R}^3$

In this section, we investigate some conditions on H and p with which the equivalence

$$\|H^{\frac{1}{4}}f\|_{L^p(\mathbb{R}^3)} \approx C_{H,p}\big\|(-\Delta)^{\frac{1}{4}}f\big\|_{L^p(\mathbb{R}^3)}$$

holds. This equivalence was studied in [7] and [4] that are of independent interest. We now introduce such an equivalence in a form for $n = 3$, which enables us to include the endpoint estimate also for that dimension.

Proposition 3.1. *Given $A \in L^2_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ and $V : \mathbb{R}^3 \to \mathbb{R}$ measurable, assume that the operators $\Delta_A = -(\nabla - iA)^2$ and $H = -\Delta_A + V$ are self-adjoint and positive on L^2 and that (1.13) holds. Moreover, assume that V_+ is of Kato class and that A and V satisfy (1.8) and*

$$|A(x)|^2 + |\nabla \cdot A(x)| + |V(x)| \le C_0 \min\left(\frac{1}{|x|^{2-\epsilon}}, \frac{1}{|x|^{2+\epsilon}}\right) \tag{3.1}$$

for some $0 < \epsilon \le 2$ and $C_0 > 0$. Then the following estimates hold:

$$\|H^{\frac{1}{4}}f\|_{L^p} \le C_{\epsilon,p}C_0\|(-\Delta)^{\frac{1}{4}}f\|_{L^p}, \quad 1 < p \le 6, \tag{3.2}$$

$$\|H^{\frac{1}{4}}f\|_{L^p} \ge C_p\|(-\Delta)^{\frac{1}{4}}f\|_{L^p}, \quad \frac{4}{3} < p < 4. \tag{3.3}$$

In showing this, we only prove (3.2) as estimate (3.3) is the same as [7, Theorem 1.2]. When $1 < p < 6$, estimate (3.2) easily follows from the Sobolev embedding theorem. However, to extend the range of p up to 6, we need a precise estimate which depends on ϵ in (3.1). Toward this, we introduce a weighted Sobolev inequality as below.

Lemma 3.2 (Theorem 1(B) in [29]). *Suppose $0 < \alpha < n$, $1 < p < q < \infty$ and $v_1(x)$ and $v_2(x)$ are nonnegative measurable functions on $\mathbb{R}^n$. Let $v_1(x)$ and $v_2(x)^{1-p'}$ satisfy the reverse doubling condition: there exist $\delta, \epsilon \in (0,1)$ such that*

$$\int_{\delta Q} v_1(x)dx \le \epsilon \int_Q v_1(x)dx \quad \text{for all cubes} \quad Q \subset \mathbb{R}^n.$$

Then the inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q v_1(x)dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n} \big|(-\Delta)^{\alpha/2}f(x)\big|^p v_2(x)dx\right)^{\frac{1}{p}}$$

holds if and only if

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q v_1(x)dx\right)^{\frac{1}{q}}\left(\int_Q v_2(x)^{1-p'}dx\right)^{\frac{1}{p'}} \le C \quad \text{for all cubes} \quad Q \subset \mathbb{R}^n.$$

From Lemma 3.2, we obtain a weighted estimate as follows.

Lemma 3.3. *Let f be a $C_0^\infty(\mathbb{R}^3)$ function, and suppose that a nonnegative weight function w satisfies*

$$w(x) \le \min\left(\frac{1}{|x|^{2-\epsilon}}, \frac{1}{|x|^{2+\epsilon}}\right) \tag{3.4}$$

for some $0 < \epsilon \le 2$. *Then, for any* $1 < p \le \frac{3}{2}$, *we have*

$$\|fw\|_{L^p} \le C_{\epsilon,p}\|\Delta f\|_{L^p}.$$

Proof. For all $1 < p < \frac{3}{2}$, we directly get

$$\left\|\frac{1}{|x|^2}f\right\|_{L^p} \le C\left\|\frac{1}{|x|^2}\right\|_{L^{\frac{3}{2},\infty}}\|f\|_{L^{\frac{3p}{3-2p},p}} \le C\|\Delta f\|_{L^p} \tag{3.5}$$

from Hölder's inequality in Lorentz spaces and the Sobolev embedding theorem. For $p = \frac{3}{2}$, by Hölder's inequality, we get

$$\left(\int_{\mathbb{R}^3}|f(x)|^{\frac{3}{2}}w(x)^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \le \left(\int_{\mathbb{R}^3}|f(x)|^q w(x)^{(1-\theta)q}dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^3}w(x)^{\frac{3q}{2q-3}\theta}dx\right)^{\frac{2q-3}{3q}}$$

for any $\frac{3}{2} < q < \infty$ and $0 < \theta < 1$. Taking $\theta = 1 - \frac{3}{2q}$, we have

$$\left(\int_{\mathbb{R}^3}|f(x)|^{\frac{3}{2}}w(x)^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \le C_{\epsilon,q}\left(\int_{\mathbb{R}^3}|f(x)|^q w(x)^{\frac{3}{2}}dx\right)^{\frac{1}{q}}$$

because of (3.4). Thus, using Lemma 3.2 with $\alpha = 2$, $(p,q) = (\frac{3}{2},q)$, $v_1(x) = w(x)^{\frac{3}{2}}$ and $v_2(x) \equiv 1$, we have

$$\left(\int_{\mathbb{R}^3}|f(x)|^{\frac{3}{2}}w(x)^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \le C_{\epsilon,q}\left(\int_{\mathbb{R}^3}|\Delta f(x)|^{\frac{3}{2}}w(x)^{\frac{3}{2}}dx\right)^{\frac{2}{3}}. \tag{3.6}$$

Combining (3.5) and (3.6), the proof is complete. □

Finally, we prove Proposition 3.1. We use Stein's interpolation theorem to the analytic family of operators $T_z = H^z \cdot (-\Delta)^{-z}$, where H^z and $(-\Delta)^{-z}$ are defined by the spectral theory. Denoting $z = x + iy$, we can decompose

$$T_z = T_{x+iy} = H^{iy}H^x(-\Delta)^{-x}(-\Delta)^{-iy}.$$

In fact, the operators H^{iy} and $(-\Delta)^{-iy}$ are bounded according to the following result.

Lemma 3.4 (Proposition 2.2 in [7]). *Consider the self-adjoint and positive operators* $-\Delta_A$ *and* $H = -\Delta_A + V$ *on* L^2. *Assume that* $A \in L^2_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ *and that the positive and negative parts* $V_\pm$ *of* V *satisfy:* V_+ *is of Kato class and*

$$\|V_-\|_K < \frac{\pi^{3/2}}{\Gamma(1/2)}.$$

Then for all $y \in \mathbb{R}$, *the imaginary powers* H^{iy} *satisfy the* $(1,1)$ *weak type estimate*

$$\|H^{iy}\|_{L^1\to L^{1,\infty}} \le C(1+|y|)^{\frac{3}{2}}.$$

Lemma 3.4 follows from the pointwise estimate for the heat kernel $p_t(x,y)$ of the operator e^{-tH} as

$$\left|p_t(x,y)\right| \le \frac{(2t)^{-3/2}}{\pi^{3/2} - \Gamma(1/2)\|V_-\|_K}e^{-\frac{|x-y|^2}{8t}}.$$

Regarding this estimate, one may refer to some references [4, 9, 30, 31].

By Lemma 3.4, we get

$$\|T_{iy}f\|_p \leq C(1+|y|)^3\|f\|_p \quad \text{for all } 1<p<\infty. \tag{3.7}$$

Then by (1.13), we have

$$\begin{aligned}\|T_{iy}f\|_{BMO_H} &:= \left\|M_H^{\#}\left(H^{iy}(-\Delta)^{-iy}f\right)\right\|_{L^\infty}\\ &\leq C(1+|y|)^{\frac{3}{2}}\left\|(-\Delta)^{-iy}f\right\|_{BMO} \leq C(1+|y|)^3\|f\|_{L^\infty},\end{aligned} \tag{3.8}$$

where

$$M_H^{\#}f(x) := \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}\left|f(y)-e^{-r^2H}f(y)\right|dy < \infty.$$

Next, consider the operator T_{1+iy}. If

$$\|H(-\Delta)^{-1}f\|_{L^p} \leq C\|f\|_{L^p} \quad \text{for all } 1<p\leq\frac{3}{2}, \tag{3.9}$$

then by (3.7), we get

$$\|T_{1+iy}f\|_{L^p} \leq C\|f\|_{L^p} \quad \text{for all } 1<p\leq\frac{3}{2}. \tag{3.10}$$

Taking $\widetilde{T}_zf := M_H^{\#}\left(T_zf\right)$ and applying (3.10) with a basic property[1]:

$$\|M_H^{\#}f\|_{L^p} \leq C\|f\|_{L^p} \quad \text{for all } 1<p\leq\infty, \tag{3.11}$$

we have

$$\|\widetilde{T}_{1+iy}f\|_{L^p} \leq C\|f\|_{L^p} \quad \text{for all } 1<p\leq\frac{3}{2}. \tag{3.12}$$

So, applying Stein's interpolation theorem to (3.8) and (3.12), we obtain

$$\|\widetilde{T}_{1/4}f\|_{L^p} \leq C\|f\|_{L^p} \quad \text{for all } 1<p\leq 6,$$

and using (3.11) again, we have

$$\|H^{1/4}f\|_{L^p} \leq C\|(-\Delta)^{1/4}f\|_{L^p} \quad \text{for all } 1<p\leq 6.$$

Now, we handle the remaining part (3.9); that is, we wish to establish the estimate

$$\|Hf\|_{L^p} \leq C\|\Delta f\|_{L^p}.$$

For a Schwartz function f, we can write

$$Hf = -\Delta f + 2iA\cdot\nabla f + (|A|^2 + i\nabla\cdot A + V)f. \tag{3.13}$$

From Hölder's inequality in Lorentz spaces and the Sobolev embedding theorem, we get

$$\|A\cdot\nabla f\|_{L^r} \leq C\|A\|_{L^{3,\infty}}\|\nabla f\|_{L^{\frac{3r}{3-r},r}} \leq C\|A\|_{L^{3,\infty}}\|\Delta f\|_{L^r}$$

for all $1<r<3$. On the other hand, applying Lemma 3.3 to (3.1), we get

$$\left\|(|A|^2 + i\nabla\cdot A + V)f\right\|_{L^r} \leq CC_0\|\Delta f\|_{L^r}$$

[1]Some properties of the BMO_L space can be found in [10].

for all $1 < r \le \frac{3}{2}$. Thus we have

$$\|Hf\|_{L^r} \le C\|\Delta f\|_{L^r} \quad \text{for all } 1 < r \le \frac{3}{2},$$

and this implies Proposition 3.1.

4. Proof of Theorem 1.3

In this final section, we prove Theorem 1.3. This part follows an argument in [7]. Let u be a solution to problem (1.1) of the magnetic Schrödinger equation in $\mathbb{R}^{n+1}$. By (3.13), we can expand H in (1.1):

$$H = -\Delta + 2iA \cdot \nabla + |A|^2 + i\nabla \cdot A + V.$$

Thus, by Duhamel's principle and the Coulomb gauge condition (1.10), we have a formal solution to (1.1) given by

$$u(x,t) = e^{itH}f(x) = e^{it\Delta}f(x) - i\int_0^t e^{i(t-s)\Delta}R(x,\nabla)e^{isH}fds, \tag{4.1}$$

where

$$R(x,\nabla) = 2iA \cdot \nabla_A - |A|^2 + V.$$

From [27] and [22] (see also (3.4) in [7]), it follows that for every admissible pair (r,q),

$$\left\| |\nabla|^{\frac{1}{2}} \int_0^t e^{i(t-s)\Delta}F(\cdot,s)ds \right\|_{L_t^q L_x^r} \le C_{n,r,q} \sum_{j\in\mathbb{Z}} 2^{j/2}\|\chi_{C_j}F\|_{L^2_{x,t}}, \tag{4.2}$$

where $C_j = \{x : 2^j \le |x| \le 2^{j+1}\}$ and χ_{C_j} is the characteristic function of the set C_j. Then, from (4.1), (1.3) and (4.2), we know that

$$\begin{aligned}\left\||\nabla|^{\frac{1}{2}}u\right\|_{L_t^qL_x^r} &\le \left\||\nabla|^{\frac{1}{2}}e^{it\Delta}f\right\|_{L_t^qL_x^r} + \left\||\nabla|^{\frac{1}{2}}\int_0^t e^{i(t-s)\Delta}R(x,\nabla)e^{isH}fds\right\|_{L_t^qL_x^r}\\ &\le C_{n,r,q}\left\||\nabla|^{1/2}f\right\|_{L_x^2} + C_{n,r,q}\sum_{j\in\mathbb{Z}}2^{j/2}\left\|\chi_{C_j}R(x,\nabla)e^{itH}f\right\|_{L^2_{x,t}}.\end{aligned}$$

For the second term in the far right-hand side, we get

$$\left\|\chi_{C_j}R(x,\nabla)e^{itH}f\right\|_{L^2_{x,t}} \le 2\left\|\chi_{C_j}A\cdot\nabla_Ae^{itH}f\right\|_{L^2_{x,t}} + \left\|\chi_{C_j}\left(|A|^2+|V|\right)e^{itH}f\right\|_{L^2_{x,t}}.$$

Next, we will use a known result in [15], which is a smoothing estimate for the magnetic Schrödinger equation.

Lemma 4.1 (Theorems 1.9 and 1.10 in [15]). *Assume $n \ge 3$, A and V satisfy conditions (1.10), (1.11), and (1.12). Then, for any solution u to (1.1) with $f \in L^2$ and $-\Delta_A f \in L^2$, the following estimate holds:*

$$\begin{aligned}&\sup_{R>0}\frac{1}{R}\int_0^\infty\int_{|x|\le R}|\nabla_Au|^2dxdt + \sup_{R>0}\frac{1}{R^2}\int_0^\infty\int_{|x|=R}|u|^2d\sigma(x)dt\\ &\quad\le C_A\|(-\Delta_A)^{\frac{1}{4}}f\|^2_{L^2}.\end{aligned}$$

From (1.9) with Lemma 4.1, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} 2^{j/2}\left\|\chi_{C_j} A\cdot\nabla_A e^{itH}f\right\|_{L^2_{x,t}}\\
&\quad\le \sum_{j\in\mathbb{Z}} 2^j\Big(\sup_{x\in C_j}|A|\Big)\Big(\frac{1}{2^{j+1}}\int_0^\infty\int_{|x|\le 2^{j+1}}|\nabla_A u|^2dxdt\Big)^{\frac12}\\
&\quad\le \Big(\sum_{j\in\mathbb{Z}} 2^j\sup_{x\in C_j}|A|\Big)\Big(\sup_{R>0}\frac1R\int_0^\infty\int_{|x|\le R}|\nabla_A u|^2dxdt\Big)^{\frac12}\\
&\quad\le C_{A,\epsilon}\left\|(-\Delta_A)^{\frac14}f\right\|_{L^2_x}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} 2^{j/2}\left\|\chi_{C_j}\big(|A|^2+|V|\big)e^{itH}f\right\|_{L^2_{x,t}}\\
&\quad\le \sum_{j\in\mathbb{Z}} 2^{j/2}\Big(\sup_{x\in C_j}\big(|A|^2+|V|\big)\Big)\Big(\int_{2^j}^{2^{j+1}} r^2\int_0^\infty\frac{1}{r^2}\int_{|x|=r}|u|^2d\sigma_r(x)dtdr\Big)^{\frac12}\\
&\quad\le \Big(\sum_{j\in\mathbb{Z}} 2^{2j}\sup_{x\in C_j}\big(|A|^2+|V|\big)\Big)\Big(\sup_{R>0}\frac{1}{R^2}\int_0^\infty\int_{|x|=R}|u|^2d\sigma_R(x)dt\Big)^{\frac12}\\
&\quad\le C_{A,V,\epsilon}\left\|(-\Delta_A)^{\frac14}f\right\|_{L^2_x}.
\end{aligned}$$

That is,

$$\left\||\nabla|^{\frac12}e^{itH}f\right\|_{L^q_tL^r_x}\le C_{n,r,q}\left\||\nabla|^{1/2}f\right\|_{L^2_x}+C_{n,r,q,A,V,\epsilon}\left\|(-\Delta_A)^{\frac14}f\right\|_{L^2_x}.$$

First, consider the case $n=3$. By (1.9), estimate (3.2) in Proposition 3.1 holds for all $1<p\le 6$. (Here, $H=-\Delta_A+V$.) Then by (3.3) in Proposition 3.1, we get

$$\begin{aligned}
\left\|H^{\frac14}e^{itH}f\right\|_{L^q_t(\mathbb{R};L^r_x(\mathbb{R}^3))}&\le C\left\||\nabla|^{\frac12}e^{itH}f\right\|_{L^q_t(\mathbb{R};L^r_x(\mathbb{R}^3))}\\
&\le C\left\||\nabla|^{\frac12}f\right\|_{L^2_x(\mathbb{R}^3)}+C\left\|(-\Delta_A)^{\frac14}f\right\|_{L^2_x(\mathbb{R}^3)}\\
&\le C\left\|H^{\frac14}f\right\|_{L^2_x(\mathbb{R}^3)}
\end{aligned}\tag{4.3}$$

for all admissible pairs (r,q). (It clearly includes the endpoint case $(n,r,q)=(3,6,2)$.)

Next, for the case $n\ge 4$, we already know that (3.2) holds for $1<p<2n$ and that (3.3) is valid for $\frac43<p<4$ under the same conditions on A and V (see [7, Theorem 1.2]). Thus, we can easily get the same bound as (4.3) for all admissible pairs (r,q).

Since the operators $H^{\frac14}$ and e^{itH} commutes, we get

$$\left\|e^{itH}f\right\|_{L^q_t(\mathbb{R};L^r_x(\mathbb{R}^n))}\le C\|f\|_{L^2_x(\mathbb{R}^n)}$$

from (4.3), and this completes the proof.

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